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Some Results on Gamma Graphs

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Abstract:

In a graph $G = (V, E)$ the set of vertices $S \subseteq V$ is a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G equals the minimum cardinality of a dominating set S in G and the set S is known as γ-set. The gamma graph of a graph G has its γ-sets as vertices and any two vertices are adjacent if the corresponding γ-sets differ exactly by one vertex. In this paper we try to include the study on the gamma graphs on corona, join and cartesian product cycles and paths.

***Keyword**: Gamma sets, Gamma graphs, Cartesian Product, Join, Corona of two graphs*

I. Introduction

Throughout this paper we consider finite simple graphs $G = (V, E)$. We use standard notations of graph theory as in Balakrishnan and Ranganathan [4]. For an introduction to the theory of domination in graphs we refer to Haynes et al. [13].

A set $S \subseteq V$ of vertices in a graph G is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. The domination number $\gamma(G)$ of G equals the minimum cardinality of a dominating set S in G and a dominating set of cardinality $\gamma(G)$is called a γ -set.

The concept of the gamma graph is introduced by Sridharan and Subramanian [1]. Let S be the collection of all γ-sets in G and the gamma graph of G, denoted by $\gamma.G$ is defined as the graph with vertex set S and any two vertices S_1 and S_2 are adjacent if $|S_1 \cap S_2| = \gamma(G) - 1$.

Sridharan and Subramanian studied the following:

i) The gamma graphs of path and cycles. [1].
ii) Every tree is γ- connected. [1].
iii) Trees and unicyclic graphs are γ-graphs [2].

Lakshmanan and Vijayakumar [3] listed the following in their work:

i) Forbidden subgraphs on five vertices of the gamma graphs.

ii) The collection of all gamma graphs is closed under the cartesian product.

iii) Some properties of cographs and their gamma graphs.

Fricke *et al* [5] have found that every tree is a gamma graph of some graph. Isaac and Bhatt [6] mentioned the inductive method of obtaining the gamma graph of cycle C_{3k+1}. They obtained $\gamma.C_{3k+1}$ is 4- regular.

In this paper we have investigated some results on gamma graphs under different graph operations such as join, corona and cartesian product of two graphs.

II. Preliminary Concepts

Definition: The **corona** product $G_1 \odot G_2$ of two graphs $G_1 = (n_1, m_1)$ and $G_2 = (V_2, E_2)$ is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and then, joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

III. Result

Definition: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. The **cartesian product** of G_1 and G_2 denoted by $G_1 \square G_2$ has vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and(v_1, v_2) are adjacent if $u_1 = v_1$ and $u_2 v_2 \in E_2$ or, $u_2 = v_2$ and $u_1 v_1 \in E_1$.

Theorem: If $G \approx C_3 \square P_n$, then $\gamma.G$ is $2n$-regular.

Proof: Let $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n\}$ be the vertices of G. Clearly $\gamma(G) = n$. Let us consider a set of vertices $S = \{u_1, u_2, \dots, u_n\}$ as a γ-set of G, and by replacing u_n by v_n and then by w_n , we can obtain new two γ-sets of G, say $S_1 = \{u_1, u_2, \dots, u_{n-1}, v_n\}$ and $S_2 =\{u_1, u_2, \dots, u_{n-1}, w_n\}$ respectively. We can repeat the procedure for each vertex of S, and it produce two different γ-sets each time. Hence, we have $2n$ different γ-sets and by the definition of gamma graphs, degree of vertex in gamma graph is $2n$. Thus, $\gamma.G$ is $2n$-regular.

Observation: The number of γ-sets in Cartesian product of C_3 and P_n is 3^n.

Definition: The **join** of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \& v \in V(G_2)\}$.

Theorem: $\gamma \cdot C_m \vee P_n$ has K_m as induced subgraph; for $n > 3$.

Proof: Let $V = \{v_1, v_2, \dots, v_m\}$ be the vertex set of C_m and $U = \{u_1, u_2, \dots, u_n\}$ be the vertex of P_n.

We know that $\gamma(C_m \vee P_n) = 2$ i.e. the γ-sets are of cardinality 2.

Let $S = \{v_j, u_i\}$; $u_i \in U$ & $v_j \in V$ be the γ-set and $|V| = m, |U| = n$.

So, we get no. of γ-sets as mn.

If $m > 6$ & $3 < n \leq 6$ then, the no. of γ-set is $mn + p$. Where, p is the no. of γ-sets of the form $\{u_i, u_j\}; i < j$.

If $n > 6$ & $3 < m \leq 6$ then, the no. of γ-set is $mn + q$. Where, q is the no. of γ-sets of the form $\{v_i, v_j\}; i < j$.

If $3 < m, n \leq 6$ then, the no. of γ-set is $mn + p + q$.

i.e. in each case we get at least mn γ-sets say, $\{v_1, u_1\}, \dots, \{v_1, u_n\}, \{v_2, u_1\}, \dots, \{v_2, u_n\}, \dots, \{v_m, u_1\}, \dots, \{v_m, u_n\}$.

For $m \geq n$ & $m < n$, $\{v_j, u_1\}: 1 \leq j \leq m$ are adjacent to each other by the definition of gamma graph. Thus, K_m is an induced subgraph in $\gamma \cdot C_m \vee P_n$.

Observations:

$\gamma \cdot C_m \vee P_1 = K_1$ for $m \geq 4$.

$\gamma \cdot C_m \vee P_2 = K_2$ for $m \geq 4$.

$\gamma \cdot C_m \vee P_3 = K_1$ for $m \geq 4$.

$\gamma \cdot C_3 \vee P_1 = K_4$.

$\gamma \cdot C_3 \vee P_2 = K_5$.

$\gamma \cdot C_3 \vee P_3 = K_4$.

$\gamma \cdot C_3 \vee P_n = K_3$ for $n > 3$.

Definition: The **corona** product $G_1 \odot G_2$ of two graphs $G_1 = (n_1, m_1)$ and $G_2 = (V_2, E_2)$ is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and then, joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

Proposition: [9] Let G be a connected graph of order m and let H be any graph of order n then $\gamma(G \odot H) = m$.

Theorem: $\gamma \cdot C_n \odot P_k = \begin{cases} 2n - regular\ graph\ , if\ k = 2 \\ n - regular\ graph\ , if\ k = 3 \\ K_1, if\ k > 3 \end{cases}$

Proof: Let $V = \{v_1, v_2, \dots, v_n\}$be the vertex set of C_n and $W = \{U_1, U_2, \dots, U_n\}$ where each U_i is a set of vertices of path. i.e. $U_i = \{u_{1i}, u_{2i}, \dots, u_{ki}\}; 1 \leq i \leq n$ because in $C_n \odot P_k$ we have n copies of P_k and $\gamma(C_n \odot P_k) = n$ by above proposition.

Let, $V = \{v_1, v_2, \dots, v_n\}$ be the γ-set of $C_n \odot P_k$.

For $k = 2$, the possible γ-sets are $S_1 = \{u_{11}, v_2, \dots, v_n\}, \dots, S_n = \{v_1, v_2, \dots, u_{1n}\}, S_{n+1} = \{u_{21}, v_2, \dots, v_n\}, \dots, S_{2n} = \{v_1, v_2, \dots, u_{2n}\}$. Each S_i's are adjacent to each other and with V by definition of gamma graph.

Therefor, $\gamma \cdot C_n \odot P_2$ is $2n$- regular.

Similarly, for $k = 3$, possible γ-sets are $S_1 = \{u_{21}, v_2, \dots, v_n\}, \dots, S_n = \{v_1, v_2, \dots, u_{2n}\}$. So, $\gamma \cdot C_n \odot P_3$ is n- regular.

For, $k > 3$ we have unique γ-set say V. Thus, $\gamma \cdot C_n \odot P_k = K_1$.

IV. Conclusion

We have taken the initiative to study the nature of gamma graph on different graph operations like cartesian product, join and corona of two graphs and observed that the gamma graph of cycle of order 3 cartesian product with path of length n and corona of cycle and path is a regular graph.

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